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Bessel's Differential Equation

In the **Sturm-Liouville Boundary Value Problem**, there is an important special case called *Bessel's Differential Equation* which arises in numerous problems, especially in polar and cylindrical coordinates. **Bessel's Differential Equation** is defined as:

$$x^2y'' + xy' + \left(x^2 - n^2\right)y = 0$$

where n is a non-negative real number. The solutions of this equation are called **Bessel Functions** of order n. Although the order n can be any real number, the scope of this section is limited to *non-negative integers*, i.e., $n = 0, 1, 2, 3, \ldots$, unless specified otherwise.

Since Bessel's differential equation is a second order ordinary differential equation, two sets of functions, the Bessel function of the first kind $J_n(x)$ and the Bessel function of the second kind (also known as the Weber Function) $Y_n(x)$, are needed to form the general solution:

$$y(x) = c_1 J_n(x) + c_2 Y_n(x)$$

However, $Y_n(x)$ is *divergent* at $x = 0$. The associated coefficient c_2 is forced to be *zero* to obtain a physically meaningful result when there is no source or sink at $x = 0$.

See plots of Bessel Functions

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Important Properties

Basic Relationship: The Bessel function of the first kind of order n can be expressed as a series of gamma functions.

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4 (2n+2)(2n+4)} - \cdots \right]$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k! \Gamma(n+k+1)}$$

The Bessel function of the second kind of order n can be expressed in terms of the Bessel function of the first kind.

$$Y_n(x) = \frac{2}{\pi} J_n(x) \left(\ln \frac{x}{2} + \gamma \right) - \frac{1}{\pi} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{m!} \left(\frac{x}{2} \right)^{2m-n}$$
$$+ \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m-1} \left[\left(1 + \frac{1}{2} + \ldots + \frac{1}{m} \right) + \left(1 + \frac{1}{2} + \ldots + \frac{1}{m+n} \right) \right]}{m!(m+n)!} \left(\frac{x}{2} \right)^{2m+n}$$
$$= \lim_{p \to n} \frac{J_p(x) \cos p\pi - J_{-p}(x)}{\sin p\pi} \quad n = 0, 1, 2, \ldots$$

Generating Function: The generating function of the Bessel Function of the first kind is

$$e^{x(t-1/t)/2} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

Recurrence Relation: A Bessel function of higher order can be expressed by Bessel functions of lower orders.

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \qquad Y_{n+1}(x) = \frac{2n}{x} Y_n(x) - Y_{n-1}(x)$$

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right] \qquad Y_n{}'(x) = \frac{1}{2}\left[Y_{n-1}(x) - Y_{n+1}(x)\right]$$

$$J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x) \qquad Y_n{}'(x) = Y_{n-1}(x) - \frac{n}{x} Y_n(x)$$

$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x) \qquad Y_n{}'(x) = \frac{n}{x} Y_n(x) - Y_{n+1}(x)$$

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x) \qquad \frac{d}{dx}\left[x^n Y_n(x)\right] = x^n Y_{n-1}(x)$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x) \qquad \frac{d}{dx}\left[x^{-n} Y_n(x)\right] = -x^{-n} Y_{n+1}(x)$$

Asymptotic Approximations: Keeping the first few terms in the series expansions, the behavior of a Bessel function at small or large x, can be captured and expressed as elementary functions which are much easier to be understood and calculated than the more abstract symbols $J_n(x)$ and $Y_n(x)$.

For small x, i.e., fixed n and $x \to 0$,

$$J_n \sim \frac{1}{2^n n!} x^n \qquad Y_0 \sim \frac{2}{\pi} \ln x \qquad Y_n \sim -\frac{2^n (n-1)!}{\pi} x^{-n}$$

For large x, i.e., fixed n and $x \gg n$,

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left[x - (2n+1)\frac{\pi}{4}\right] \qquad Y_n(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left[x - (2n+1)\frac{\pi}{4}\right]$$

Orthogonality: Suppose that λ_{np} is the $p\text{th}$ non-negative root of the $n\text{th}$ characteristic equation

$$A_n J_n\left(\frac{x}{\rho}\right) + B_n x J_n{}'\left(\frac{x}{\rho}\right) = 0$$

associated with a physical problem defined on the interval of $0 \le x \le \rho$. It can be verified that

$$\int_0^1 xJ_n(\lambda_{np}\frac{x}{\rho})J_n(\lambda_{nq}\frac{x}{\rho})dx = 0 \quad p \neq q$$

By using this orthogonality, the $n\text{th}$ component of the general solution of the physical problem is

$$y_n(x) = \sum_{p=0}^{\infty} c_{np} J_n(\lambda_{np}\frac{x}{\rho})$$

The general solution thus yields

$$y(x) = \sum_{n=0}^{\infty}\sum_{p=1}^{\infty} c_{np} J_n(\lambda_{np}\frac{x}{\rho})$$

where

$$c_{np} = \frac{\int_0^{\rho} xf(x)J_n(\lambda_{np}\frac{x}{\rho})dx}{\int_0^{\rho} x\left[J_n(\lambda_{np}\frac{x}{\rho})\right]^2 dx}$$

and $f(x)$ is a piecewise continuous function, generally the non-homogeneous term of the problem.

This orthogonal series expansion is also known as a **Fourier-Bessel Series** expansion or a **Generalized Fourier Series** expansion. The transform based on this relationship is called a **Hankel Transform**.

Hankel Function: Similar to $e^{\pm ikx} = \cos kx \pm i \sin kx$, the Hankel function of the first kind and second kind, prominent in the theory of wave propagation, are defined as

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x) \qquad H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x)$$

For large x, i.e., fixed n and $x \gg n$,

$$H_n^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}} e^{i(x-\frac{\pi}{4}-\frac{n\pi}{2})} \qquad H_n^{(2)}(x) \sim \sqrt{\frac{2}{\pi x}} e^{-i(x-\frac{\pi}{4}-\frac{n\pi}{2})}$$

Modified Bessel Function: Similar to the relations between the trigonometric functions and the hyperbolic trigonometric functions,

$$\sin ix = i \sinh x \qquad \cos ix = \cosh x \qquad e^{\pm x} = \cosh x \pm i \sinh x$$

The modified Bessel functions of the first and second kind of order n are defined as

$$I_n(x) = i^{-n} J_n(ix) \text{ and } K_n(x) = \frac{\pi}{2} i^{n+1} \left[J_n(ix) + i Y_n(ix) \right] = \frac{\pi}{2} i^{n+1} H_n^{(1)}(ix)$$

See further detail on the modified Bessel functions.

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Special Results

$$J_\nu(0) = \begin{cases} 1 & \nu = 0 \\ 0 & \nu > 0 \\ 0 & \nu = -1, -2, -3, \ldots \\ \infty & \nu < 0,\ \nu \neq -1, -2, \ldots \end{cases} \qquad Y_\nu(0) = -\infty$$

$$J_{-n}(x) = (-1)^n J_n(x) \qquad J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(n\theta - x \sin\theta) d\theta$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right) \qquad J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\cos x}{x} + \sin x \right)$$

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\left(\frac{3}{x^2} - 1 \right) \sin x - \frac{3}{x} \cos x \right] \qquad J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{3}{x} \sin x + \left(\frac{3}{x^2} - 1 \right) \cos x \right]$$

$$J_n(x) Y_n'(x) - J_n'(x) Y_n(x) = \frac{2}{\pi x} \qquad J_\nu(x) J_{-\nu}'(x) - J_\nu'(x) J_{-\nu}(x) = -\frac{2 \sin \nu x}{\pi x}$$

$$\sin(x \sin\theta) = 2 \sum_{n=1}^{\infty} J_{2n-1}(x) \sin\left[(2n-1)\theta\right] \qquad \cos(x \sin\theta) = J_0(x) + 2 \sum_{n=1}^{\infty} J_{2n}(x) \cos 2n\theta$$

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The Modified Bessel's Differential Equation

Similar to the relations between the trigonometric functions and the hyperbolic trigonometric functions,

$$\sin ix = i \sinh x \qquad \cos ix = \cosh x \qquad e^{\pm x} = \cosh x \pm i \sinh x$$

the modified Bessel functions of the first kind of order n, $I_n(x)$, can be expressed by the Bessel function of the first kind

$$J_n(ix) = i^n I_n(x)$$

The **modified Bessel's differential equation** is defined in a similar manner by changing the variable x to ix in Bessel's differential equation:

$$x^2 y'' + xy' - \left(x^2 + n^2\right) y = 0$$

Its general solution is

$$y(x) = c_1 I_n(x) + c_2 K_n(x)$$

where

$$I_n(x) = i^{-n} J_n(ix)$$

$$K_n(x) = \frac{\pi}{2} i^{n+1} \left[J_n(ix) + iY_n(ix) \right] = \frac{\pi}{2} i^{n+1} H_n^{(1)}(ix)$$
$$= \lim_{p \to n} \frac{\pi}{2} \left[\frac{I_{-p}(x) - I_p(x)}{\sin p\pi} \right]$$

are the **modified Bessel functions** of the first and second kind respectively.

See plots of Modified Bessel Functions

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Important Properties

Generating Function: The generating function of the Bessel Function of the first kind is

$$e^{x(t+1/t)/2} \sum_{n=-\infty}^{\infty} I_n(x) t^n$$

Recurrence Relation: A modified Bessel function of higher order can be expressed by modified Bessel functions of lower orders.

$$I_{n+1}(x) = I_{n-1}(x) - \frac{2n}{x} I_n(x) \qquad K_{n+1}(x) = K_{n-1}(x) + \frac{2n}{x} K_n(x)$$

$$I_n'(x) = \frac{1}{2}\left[I_{n-1}(x) + I_{n+1}(x)\right] \qquad K_n'(x) = -\frac{1}{2}\left[K_{n-1}(x) + K_{n+1}(x)\right]$$

$$I_n'(x) = I_{n-1}(x) - \frac{n}{x} I_n(x) \qquad K_n'(x) = -K_{n-1}(x) - \frac{n}{x} K_n(x)$$

$$I_n'(x) = \frac{n}{x} I_n(x) + I_{n+1}(x) \qquad K_n'(x) = \frac{n}{x} K_n(x) - K_{n+1}(x)$$

$$\frac{d}{dx}\left[x^n I_n(x)\right] = x^n I_{n-1}(x) \qquad \frac{d}{dx}\left[x^n K_n(x)\right] = -x^n K_{n-1}(x)$$

$$\frac{d}{dx}\left[x^{-n}I_n(x)\right]=x^{-n}I_{n+1}(x) \qquad \frac{d}{dx}\left[x^{-n}K_n(x)\right]=-x^{-n}K_{n+1}(x)$$

Asymptotic Approximations:

For large x, i.e., fixed n and $x \gg n$,

$$I_n(x)\sim\frac{e^x}{\sqrt{2\pi x}} \qquad K_n(x)\sim\frac{e^{-x}}{\sqrt{\frac{2}{\pi}x}}$$

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Special Results

$$I_\nu(0)=\begin{cases}1 & \nu=0\\ 0 & \nu>0\\ 0 & \nu=-1,-2,-3,\ldots\\ \infty_c & \nu<0,\ \nu\neq-1,-2,\ldots\end{cases} \qquad K_\nu(0)=\begin{cases}\infty & \nu=0\\ \infty_c & \text{otherwise}\end{cases}$$

where ∞_c is complex infinity.

$$I_{-n}(x)=I_n(x) \qquad K_{-n}(x)=K_n(x)$$

$$I_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sinh x \qquad I_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cosh x$$

$$I_{3/2}(x)=\sqrt{\frac{2}{\pi x}}\left(\cosh x-\frac{\sinh x}{x}\right) \qquad I_{-3/2}(x)=\sqrt{\frac{2}{\pi x}}\left(\sinh x-\frac{\cosh x}{x}\right)$$

$$I_{5/2}(x)=\sqrt{\frac{2}{\pi x}}\left\{\left(\frac{3}{x^2}+1\right)\sinh x-\frac{3}{x}\cosh x\right\} \qquad I_{-5/2}(x)=\sqrt{\frac{2}{\pi x}}\left\{\left(\frac{3}{x^2}+1\right)\cosh x-\frac{3}{x}\sinh x\right\}$$

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