The Partial Differential Equation Toolbox 1

for use with MATLAB®

The MATH WORKS Inc.

The Partial Differential Equation (PDE) Toolbox extends the MATLAB technical computing environment with powerful and flexible tools for the study and solution of PDEs in two-space dimensions (2-D) and time. The PDE Toolbox provides a set of command line functions and a graphical user interface for preprocessing, solving, and postprocessing generic 2-D PDEs using the Finite Element Method (FEM). The toolbox also provides automatic and adaptive meshing capabilities and a suite of eight application modes for common PDE application areas such as heat transfer.

BACKGROUND

PDEs are used as mathematical models for phenomena in all branches of engineering and science. For example, the elliptic and parabolic equations can be used for steady and unsteady heat transfer in solids, for flow in porous media and diffusion problems, for steady electrostatics of dielectric and conductive media, and for potential flow. The hyperbolic PDE is used for transient and harmonic wave propagation in acoustics and electromagnetics, and for transverse motions of membranes. The elliptic system of PDEs can be used to solve plane stress and plane strain problems in structural mechanics.
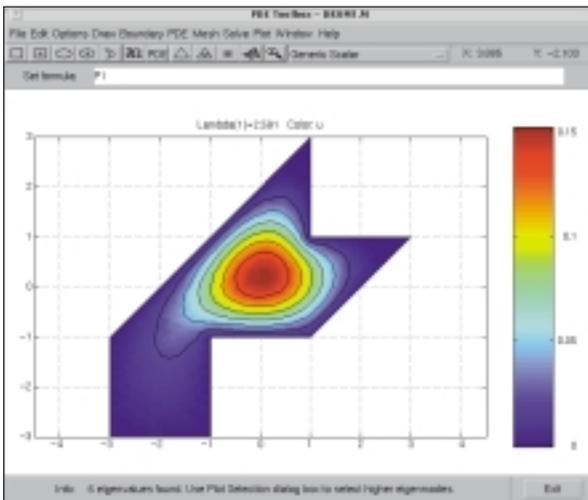
First eigenfunction of a drum membrane.

PARTIAL DIFFERENTIAL EQUATIONS

The basic scalar equation of the toolbox is the elliptic PDE

$$-\nabla \cdot (c\nabla u) + au = f \text{ in } \Omega$$

where ∇ is the vector $(\partial/\partial x, \partial/\partial y)$, and c is a 2-by-2 matrix function on Ω, the bounded planar domain of interest. c, a, and f can be complex valued functions of x and y.

The toolbox can also handle the parabolic PDE

$$d\frac{\partial u}{\partial t} - \nabla \cdot (c\nabla u) + au = f$$

the hyperbolic PDE

$$d\frac{\partial^2 u}{\partial t^2} - \nabla \cdot (c\nabla u) + au = f$$

and the eigenvalue PDE

$$-\nabla \cdot (c\nabla u) + au = \lambda du$$

where d is a complex valued function on Ω and λ is the eigenvalue. For parabolic and hyperbolic PDEs, c, a, f, and d can be complex valued functions of x, y, and t. A nonlinear Newton solver is available for the nonlinear elliptic PDE

$$-\nabla \cdot (c(u)\nabla u) + a(u)u = f(u)$$

where the coefficients defining c, a, and f can be functions of x, y, and the unknown solution u. All solvers can handle the PDE system with multiple dependent variables

$$-\nabla \cdot (c_{11}\nabla u) - \nabla \cdot (c_{12}\nabla v) + a_{11}u + a_{12}v = f_1$$
$$-\nabla \cdot (c_{21}\nabla u) - \nabla \cdot (c_{22}\nabla v) + a_{21}u + a_{22}v = f_2$$

Systems of dimension two can be handled from the interactive user interface. From the command line, an arbitrary number of dimensions can be handled. There is also an adaptive mesh refinement algorithm for the elliptic and nonlinear elliptic PDE problems.

BOUNDARY CONDITIONS

The following boundary conditions can be handled for scalar u

- Dirichlet: $hu = r$ on the boundary $\partial\Omega$
- Generalized Neumann: $\vec{n} \cdot (c\nabla u) + qu = g$ on $\partial\Omega$

where $\vec{n}$ is the outward unit normal and g, q, h, and r can be complex valued functions of x and y defined on $\partial\Omega$. For the nonlinear PDE, the coefficients may depend on u, and for time-dependent problems the coefficients may also depend on t. For PDE systems, Dirichlet, generalized Neumann, and mixed boundary conditions are supported.

PDE TOOLBOX APPLICATION MODES
- Structural Mechanics - Plane Stress
- Structural Mechanics - Plane Strain
- Electrostatics
- Magnetostatics
- AC Power Electromagnetics
- Conductive Media DC
- Heat Transfer
- Diffusion
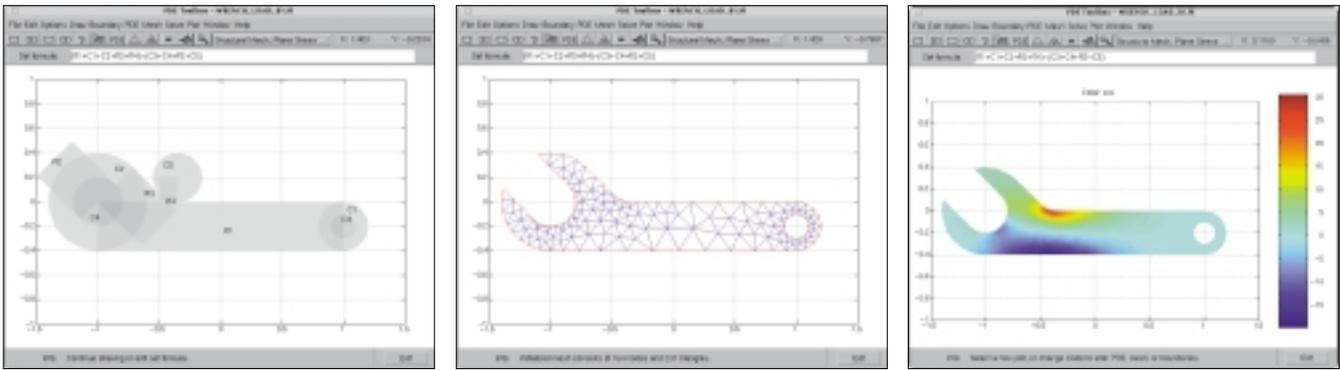
The toolbox and graphical user interface provide a comprehensive tool for creating, solving, and visualizing PDE problems. Here the complex geometry of a wrench is defined using basic geometric objects, automeshed, and analyzed for a given load configuration.

FEATURES

The graphical user interface and command line functionality of the toolbox have been designed to intuitively follow the process of solving PDEs. The PDE solution process using the Finite Element Method can be characterized by six generic steps.

These six steps and corresponding PDE Toolbox usage modes are: geometry definition (draw mode), specification of boundary conditions (boundary mode), selection of PDE coefficients that define the problem (PDE mode), finite element discretization (mesh mode), specification of initial conditions and solution of the PDE (solve mode), and postprocessing of the solution (plot mode). Each of these modes is available through the command line or graphical user interface.

- In *draw mode*, you can create Ω, the geometry, using the constructive solid geometry (CSG) model paradigm. A set of solid building blocks (square, rectangle. circle, ellipse, and polygon) are provided. These blocks can be combined in flexible ways to define complex geometries. The original shapes of the solid building blocks constitute borders between subdomains of the model.

- In *boundary mode*, you can specify the boundary conditions. You can have different types of boundary conditions on different boundaries, or remove subdomain borders if desired.

- In *PDE mode*, you can interactively specify the type of PDE problem, as well as the coefficients *c, a, f,* and *d*. You can specify the coefficients for each subdomain independently, e.g., to represent different material properties.

- In *mesh mode*, you can control the fully automated mesh generation and refinement process.

- In *solve mode*, you can invoke and control the nonlinear and adaptive solver for elliptic problems. For parabolic and hyperbolic PDE problems, you can specify the initial values and obtain solutions at specific times. For the eigenvalue solver, you can define the interval over which to search for eigenvalues.

- In *plot mode*, surface, mesh, contour, and other plot types are available. You can simultaneously visualize multiple solution properties using color, height, and ector fields. The FEM-mesh can be overlaid on all plots and can even be shown in the displaced position. For parabolic and hyperbolic equations, you can animate the solution as it changes with time.

DOCUMENTATION

The *PDE Toolbox User's Guide* describes the toolbox functionality in a thorough tutorial and reference format. Numerous examples of PDEs from science and engineering are solved using both the command line and graphical user interface. This facilitates the use of the toolbox in research, engineering applications, and educational programs.

OPEN AND EXTENSIBLE

The PDE Toolbox has been implemented completely in M-files that give you complete access to the source code and algorithms. The open-systems philosophy of MATLAB and the toolboxes enables you to make changes to existing functions or add your own.

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